Step 6 Final deduction

Theorem 6.33 allows us to prove the Prime Number Theorem with an error term.

Theorem 6.34 ***Prime Number Theorem with error term.***

$$\psi(x) = x + O\left(x \exp\left(-c \log^{1/10} x\right)\right),$$

for some $c > 0$.

This will follow from

Lemma 6.35 *If*

$$\int_1^x \psi(t)\, dt = \frac{1}{2}x^2 + x^2 \mathcal{E}(x)\,, \tag{44}$$

then there exist a constants $C > 0$ such that

$$|\psi(x) - x| \le C\tau(x)\, x,$$

where

$$\tau^2(x) = \max_{x/2 \le t \le 3x/2} |\mathcal{E}(t)|\,.$$

Proof The important observation is that $\psi(x)$ is an *increasing function of* x. Then, with $h = h\,(x)$ to be chosen,

$$\frac{1}{h}\int_{x-h}^x \psi(t)\, dt \le \psi(x) \le \frac{1}{h}\int_x^{x+h} \psi(t)\, dt. \tag{45}$$

The right hand side here equals

$$\frac{1}{h}\int_x^{x+h} \psi(t)\, dt = \frac{1}{h}\left(\int_1^{x+h} \psi(t)\, dt - \int_1^x \psi(t)\, dt\right) \tag{46}$$

The main term which comes from using (44) within (46) equals

$$\frac{1}{h}\left(\frac{(x+h)^2}{2} - \frac{x^2}{2}\right) = x + \frac{h}{2}.$$

The error from using (44) is

$$\begin{aligned}\frac{1}{h}\left|(x{+}h)^2\,\mathcal{E}(x{+}h)-x^2\mathcal{E}(x)\right| &\le \frac{1}{h}\left(\left|(x{+}h)^2\,\mathcal{E}(x{+}h)\right|+\left|x^2\mathcal{E}(x)\right|\right)\\ &\le 2\frac{(x{+}h)^2}{h}\max_{x<t<x+h}|\mathcal{E}(t)|\,,\end{aligned}$$

the factor 2 arising from (44) having been used twice within (46). For simplicity assume that $h<x/2$, so this error is

$$\le \frac{9x^2}{2h}\max_{x\le t\le x+3x/2}|\mathcal{E}(t)|\le \frac{9x^2}{2h}\tau^2(x)\,.$$

Similarly, the left hand side of (45) equals

$$\frac{1}{h}\left(\int_1^x\psi(t)\,dt-\int_1^{x-h}\psi(t)\,dt\right)$$

which, by (44), equals $x-h/2$ with error

$$\le 2\frac{x^2}{h}\max_{x/2\le t\le x}|\mathcal{E}(t)|\le 2\frac{x^2}{h}\tau^2(x)\,.$$

The bounds above can be combined as

$$x-\frac{h}{2}-\frac{2x^2}{h}\tau^2(x)\le\psi\left(x\right)\le x+\frac{h}{2}+\frac{9x^2}{2h}\tau^2(x)\,.$$

Choose $h\left(x\right)=x\tau(x)$ (which 'balances' the error terms) to obtain

$$x-\frac{5}{2}x\tau(x)\le\psi\left(x\right)\le x+5x\tau(x)\,,$$

which gives the stated result with $C=5$.

End of proof of Lemma 6.35

Proof of PNT with error. From Theorem 6.34 we can choose

$$\mathcal{E}(t)=\exp\left(-c\log^{1/10}x\right)$$

in which case $\tau(x)=\exp\left(-\left(c/2\right)\log^{1/10}\left(x/2\right)\right)$ which is of the form

$$O\left(\exp\left(-c\log^{1/10}x\right)\right),$$

where c is a constant *that need not be the same at each occurrence.*

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Note the best error to date in the Prime Number Theorem, due to Walfisz, 1963, (and so 50 years old) is

$$\psi(x) = x + O\left(x \exp\left(-c\frac{(\log x)^{3/5}}{(\log\log x)^{1/5}}\right)\right).$$

Theorem 6.36 ***Prime Number Theorem with error term.***

$$\pi(x) = \mathrm{li}x + O\left(x \exp\left(-c \log^{1/10} x\right)\right),$$

for some $c > 0$*, where*

$$\mathrm{li}x = \int_2^x \frac{dt}{\log t}.$$

Proof We have seen by Partial Summation that

$$\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \theta(t) \frac{dt}{t \log^2 t}.$$

If $\theta(x) = x + \mathcal{E}(x)$ then

$$\pi(x) = \frac{x + \mathcal{E}(x)}{\log x} + \int_2^x (t + \mathcal{E}(t)) \frac{dt}{t \log^2 t}.$$

Using integration by parts on the main terms gives, by first integrating the $1/t \log^2 t$ factor,

$$\frac{x}{\log x} + \int_2^x t \frac{dt}{t \log^2 t} = \frac{x}{\log x} + \left[-\frac{t}{\log t}\right]_2^x + \int_2^x \frac{dt}{\log t} = \mathrm{li}x + \frac{2}{\log 2}.$$

Hence

$$\pi(x) = \mathrm{li}x + \frac{2}{\log 2} + \frac{\mathcal{E}(x)}{\log x} + \int_2^x \mathcal{E}(t) \frac{dt}{t \log^2 t}. \qquad (47)$$

What is $\mathcal{E}(x)$? Combining $\theta(x) = \psi(x) + O\left(x^{1/2}\right)$ from the notes with Theorem 6.34 above gives

$$\mathcal{E}(x) \ll x^{1/2} + x \exp\left(-c \log^{1/10} x\right) \ll x \exp\left(-c \log^{1/10} x\right).$$

For the integral term in (47) the integral can be split at $\sqrt{x}$ to prove

$$\int_2^x \mathcal{E}(t)\,\frac{dt}{t\log^2 t} \ll x\exp\left(-c\log^{1/10} x\right),$$

though with a different c to that in Theorem 6.34. ∎